

SLICE REGULAR BESOV SPACES OF HYPERHOLOMORPHIC FUNCTIONS AND COMPOSITION OPERATORS

SANJAY KUMAR AND KHALID MANZOOR

ABSTRACT. In this paper we investigate some basic results on the slice regular Besov spaces of hyperholomorphic functions on the unit ball \mathbb{B} . We also characterize the boundedness, compactness and find the essential norm estimates of composition operators between these spaces.

1. INTRODUCTION

In the last ten years the theory of slice regular functions is developed systemically in the papers [22, 30, 31, 32, 34, 35, 36, 37, 38, 39, 40, 41]. Slice hyperholomorphic functions when defined and takes values in quaternions are called slice regular, see [5, 6, 28]. In case they are defined on the Euclidean space \mathbb{R}^{N+1} and takes values in the Clifford algebra \mathbb{R}_ω they are called slice monogenic functions, see [15, 16]. Several function spaces of the slice hyperholomorphic functions are studied. The quaternionic Hardy spaces are studied in [6, 7, 8, 12, 13, 14, 51]. The Bergman spaces of slice hyperholomorphic functions are investigated in [19, 20, 21]. For Fock spaces in the slice hyperholomorphic settings, see [5]. Further, weighted Bergman spaces, Bloch, Besov and Dirichlet spaces of slice hyperholomorphic functions on the unit ball are considered in [47]. D. Alpay etc.al studied Schur analysis in the slice hyperholomorphic setting see e.g., [1, 2, 6, 8] and references therein. The study of slice hyperholomorphic functions have wide range of applications. For complete discussion of slice regular functions and their applications, we refer the book [33] and a recent survey [18]. For each $q \in \mathbb{H}$, we can write $\mathbb{H} = \{q = x_0 + ix_1 + jx_2 + kx_3, \text{ for all } x_1, x_2, x_3 \in \mathbb{R}\}$, where $\{1, i, j, k\}$ form the basis of quaternions with imaginary units i, j, k such that $i^2 = j^2 = k^2 = -1, ij = -ji = k, jk = -kj = i, ki = -ik = j$. The Euclidean norm on \mathbb{H} is given by $|q| = \sqrt{q\bar{q}} = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2}$, where $\bar{q} = Re(q) - Im(q) = x_0 - (ix_1 + jx_2 + kx_3)$ represents the conjugate of q with $Re(q) = x_0, Im(q) = ix_1 + jx_2 + kx_3$ and the multiplicative inverse q^{-1} of non-zero quaternion q is given by $\frac{\bar{q}}{|q|^2}$. An element q in \mathbb{H} can be also written as linear combination of two complex numbers $q = (x_0 + ix_1) + (x_2 + ix_3)j$. By symbol \mathbb{S} we denote the two-dimensional unit sphere of purely imaginary quaternions i.e, $\mathbb{S} = \{q = ix_1 + jx_2 + kx_3 \text{ such that } x_1^2 + x_2^2 + x_3^2 = 1\}$. If $I \in \mathbb{S}$ then $I^2 = -1$. Let $\mathbb{C}(i)$ be

Date: September 9, 2016.

2000 Mathematics Subject Classification. Primary 47B38, 47B33, 30D55.

Key words and phrases. Besov space, slice hyperholomorphic functions, Slice Regular Besov space, composition operators, \mathbb{H} -valued p-carleson measure.

the space generated by $\{1, i\}$. For any $i \in \mathbb{S}$, let $\Omega_i = \Omega \cap \mathbb{C}_i$, for some subset (domain) Ω of \mathbb{H} . For a nonreal quaternion q we can write $q = x + Im(q) = x + I_q |Im(q)| = x + I_q y$, where $x = x_0$, $y = ix_1 + jx_2 + kx_3$ with $I_q = \frac{Im(q)}{|Im(q)|}$, so it lies in the complex plane $\mathbb{C}(i)$. We define the slice regular functions on the open ball $\mathbb{B}(0, 1) = \mathbb{B} = \{q \in \mathbb{H} : |q| < 1\}$ in \mathbb{H} and $\mathbb{B} \cap \mathbb{C}(i) = \mathbb{B}_i$ denote the unit disk in the complex plane, for $i \in \mathbb{S}$. The study of slice holomorphic functions is now an active area of research and lot of work is being done in this direction. For slice holomorphic functions we refer to [23, 26, 40, 47] and references therein. Here, we collect some basic definitions and basic results already obtained in the quaternionic-valued slice regular functions.

Definition 1.1. *Let Ω be an open set in \mathbb{H} . A real differential function $f : \Omega \rightarrow \mathbb{H}$ is said to be (left) slice regular or slice hyperholomorphic on Ω_i , if for every $i \in \mathbb{S}$,*

$$\frac{\partial}{\partial x} f_i(x + iy) + i \frac{\partial}{\partial y} f_i(x + iy) = 0,$$

where f_i denote the restriction of f to $\Omega \cap \mathbb{C}(i)$. The class of slice regular function on Ω is denoted by $SR(\Omega)$.

Lemma 1.1. [26, Lemma 4.1.7](*Splitting Lemma*) *If f is a slice regular function on the domain Ω , then for any $i, j \in \mathbb{S}$, with $i \perp j$ there exists two holomorphic functions $f_1, f_2 : \Omega_i \rightarrow \mathbb{C}(i)$ such that*

$$(1) \quad f_i(z) = f_1(z) + f_2(z)j; \text{ for any } z = x + iy \in \Omega_i.$$

One of the most important property of the slice regular functions is their Representation Formula. It only holds on the open sets which are stated below.

Definition 1.2. *Let Ω be an open set in \mathbb{H} . We say Ω is Axially symmetric if for any $q = x + I_q y \in \Omega$, all the elements $x + iy$ is contained in Ω , for all $i \in \mathbb{S}$ and Ω is said to be slice domain if $\Omega \cap \mathbb{R}$ is non empty and $\Omega \cap \mathbb{C}(i)$ is a domain in $\mathbb{C}(i)$ for all $i \in \mathbb{S}$.*

Theorem 1.3. [26, Theorem 4.3.2] (*Representation Formula*) *Let f be a slice regular function in the symmetric slice domain $\Omega \subseteq \mathbb{H}$ and let $j \in \mathbb{S}$. Then for all $z = x + iy \in \Omega$ with $i \in \mathbb{S}$, the following equality holds*

$$f(x + iy) = \frac{1}{2} \{ (1 - ij)f(x + jy) + (1 + ij)f(x - jy) \}.$$

Remark 1.4. *Let i, j be orthogonal imaginary units in S and Ω be an axillary symmetric slice domain. Then the Splitting Lemma and the Representation formula generate a class of operators on the slice regular functions as follows:*

$$Q_i : SR(\Omega) \rightarrow hol(\Omega_i) + hol(\Omega_i)j$$

$$Q_i : f \mapsto f_1 + f_2j$$

$$P_i : hol(\Omega_i) + hol(\Omega_i)j \rightarrow SR(\Omega)$$

$$P_i[f](q) = P_i[f](x + I_q y) = \frac{1}{2}[(1 - I_q i)f(x + iy) + (1 + I_q i)f(x - iy)].$$

Also,

$$P_i \circ Q_i = I_{SR(\Omega)} \text{ and } Q_i \circ P_i = I_{SR(hol(\Omega_i) + hol(\Omega_i))},$$

where I is an identity operator.

Since pointwise product of functions does not preserve slice regularity, a new multiplication operation for regular functions is defined. In the special case of power series, the regular product (or \star -product) of $f(q) = \sum_{n=0}^{\infty} q^n a_n$ and $g(q) = \sum_{n=0}^{\infty} q^n b_n$ is

$$f \star g(q) = \sum_{n=0}^{\infty} q^n \sum_{k=0}^n a_k b_{n-k}.$$

The \star -product is related to the standard pointwise product by the following formula.

Theorem 1.5. [13, Proposition 2.4] *Let f, g be regular functions on \mathbb{B} . Then $f \star g(q) = 0$ if $f(q) = 0$ and $f(q)g(f(q)^{-1}qf(q))$ if $f(q) \neq 0$. The reciprocal $f^{-\star}$ of a regular function $f(q) = \sum_{n=0}^{\infty} q^n a_n$ with respect to the \star -product is*

$$f^{\star}(q) = \frac{1}{f \star f^c(q)} f^c(q),$$

where $f^c(q) = \sum_{n=0}^{\infty} q^n \overline{a_n}$ is the regular conjugate of f . The function $f^{-\star}$ is regular on $\mathbb{B} \setminus \{q \in \mathbb{B} | f \star f^c(q) = 0\}$ and $f \star f^{-\star} = 1$ there.

2. BESOV SPACES

Now we define Besov space of slice hyperholomorphic functions on the unit ball \mathbb{B} . Let \mathbb{D} be a unit disk in the complex plane \mathbb{C} and dA be the normalized area measure on \mathbb{D} . For $1 < p < \infty$, a holomorphic function $f : \mathbb{D} \rightarrow \mathbb{C}$ is said to be in Besov space $\mathfrak{B}_{p,\mathbb{C}}(\mathbb{D})$ if

$$\int_{\mathbb{D}} |(1 - |z|^2)f'(z)|^p d\lambda(z) < \infty,$$

where $d\lambda(z) = \frac{dA(z)}{(1 - |z|^2)^2}$ and is Möbius invariant measure on \mathbb{D} . The space $\mathfrak{B}_{p,\mathbb{C}}$ is a Banach space under the norm

$$\|f\|_{\mathfrak{B}_{p,\mathbb{C}}} = |f(0)| + \left(\int_{\mathbb{D}} |(1 - |z|^2)f'(z)|^p d\lambda(z) \right)^{\frac{1}{p}}.$$

For details on the Besov space of the unit disk one can refer to [11, 58, 59] and references therein.

Definition 2.1. *Let $p > 1$ and let $i \in \mathbb{S}$. The quaternionic right linear space of slice regular functions f is said to be the quaternionic slice regular Besov space on the unit ball \mathbb{B} , if*

$$\sup_{i \in \mathbb{S}} \int_{\mathbb{B}_i} \left| (1 - |q|^2) \frac{\partial f}{\partial x_0}(q) \right|^p d\lambda_i(q) < \infty,$$

that is,

$$\mathfrak{B}_p = \{f \in SR(\mathbb{B}) : \sup_{i \in \mathbb{S}} \int_{\mathbb{B}_i} \left| (1 - |q|^2) \frac{\partial f}{\partial x_0}(q) \right|^p d\lambda_i(q) < \infty\},$$

where $d\lambda_i(q) = \frac{dA_i(q)}{(1-|q|^2)^2}$ and is Möbius invariant measure on \mathbb{B} . The space \mathfrak{B}_p is a Banach space under the norm

$$\|f\|_{\mathfrak{B}_p} = |f(0)| + \left(\sup_{i \in \mathbb{S}} \int_{\mathbb{B}_i} \left| (1-|q|^2) \frac{\partial f}{\partial x_0}(q) \right|^p d\lambda_i(q) \right)^{\frac{1}{p}}.$$

For details on Besov spaces of quaternions holomorphic functions one can refer to [47]. By space $\mathfrak{B}_{p,i}$, $p > 1$, we mean the quaternionic right linear space of slice regular functions on the unit ball \mathbb{B} such that

$$\int_{\mathbb{B}_i} |(1-|z|^2)Q_i[f]'(z)|^p d\lambda_i(z) < \infty,$$

and the norm of this space is given by

$$\|f\|_{\mathfrak{B}_{p,i}} = |f(0)| + \left(\int_{\mathbb{B}_i} |(1-|z|^2)Q_i[f]'(z)|^p d\lambda_i(z) \right)^{\frac{1}{p}}$$

where $Q_i[f]'(z) = \frac{\partial Q_i[f]}{\partial x_0}(z)$ is a holomorphic map of complex variable $z = x_0 + iy$ and $i \in \mathbb{S}$.

Remark 2.2. Let $j \in \mathbb{S}$ be such that $j \perp i$. Then there exist holomorphic functions $f_1, f_2 : \mathbb{B}_i \rightarrow \mathbb{C}(i)$ such that $Q_i[f] = f_1 + f_2 j$ and so $\frac{\partial f}{\partial x_0}(z) = f_1'(z) + f_2'(z)$ for some $z \in \mathbb{B}_i$. Thus, for $z \in \mathbb{B}_i$, it follows that

$$|f_l'(z)|^p \leq \left| \frac{\partial f}{\partial x_0}(z) \right|^p \leq 2^{\max(0, p-1)} (|f_1'(z)|^p + |f_2'(z)|^p), \quad l = 1, 2.$$

Thus, the function $f \in \mathfrak{B}_{p,i}$ if and only if $f_1, f_2 \in \mathfrak{B}_{p,\mathbb{C}}$ on \mathbb{B}_i , (see [47, Remark 4.3]).

The proof of the following proposition is analogous to [47, Proposition 2.6].

Proposition 2.3. Let $i \in \mathbb{S}$, then $f \in \mathfrak{B}_{p,i}$, $p > 1$ if and only if $f \in \mathfrak{B}_p$. Moreover, the spaces $(\mathfrak{B}_{p,i}, \|\cdot\|_{\mathfrak{B}_{p,i}})$ and $(\mathfrak{B}_p, \|\cdot\|_{\mathfrak{B}_p})$ have equivalent norms. More precisely, one has

$$\|f\|_{\mathfrak{B}_{p,i}}^p \leq \|f\|_{\mathfrak{B}_p}^p \leq 2^p \|f\|_{\mathfrak{B}_{p,i}}^p.$$

For all $z, w \in \mathbb{D}$, Bergman metric on the unit disc \mathbb{D} in the complex plane \mathbb{C} is given by

$$\beta(z, w) = \frac{1}{2} \log \frac{1 + \rho(z, w)}{1 - \rho(z, w)},$$

where $\rho(z, w) = \left| \frac{z-w}{1-\bar{z}w} \right|$.

Definition 2.4. [47]. For $i \in \mathbb{S}$ and all $z, w \in \mathbb{B}_i$, we define

$$\beta_i(z, w) = \frac{1}{2} \log \left(\frac{1 + \frac{|z-w|}{|1-\bar{z}w|}}{1 - \frac{|z-w|}{|1-\bar{z}w|}} \right).$$

Proposition 2.5. *For $1 < p, t < \infty$, with $\frac{1}{p} + \frac{1}{t} = 1$. Let $f \in \mathfrak{B}_p$ and $i \in \mathbb{S}$ be fixed. Then for all $q, w \in \mathbb{B}_i$, there exists a constant $M_p > 0$ such that*

$$|f(q) - f(w)| \leq 2M_p \|f\|_{\mathfrak{B}_p} \beta_i(q, w)^{\frac{1}{t}},$$

where

$$\beta_i(q, w) = \frac{1}{2} \log \left(\frac{1 + \frac{|q-w|}{|1-\bar{q}w|}}{1 - \frac{|q-w|}{|1-\bar{q}w|}} \right).$$

Proof. By Lemma 1.1, there exist two holomorphic functions $f_1, f_2 : \mathbb{B}_i \rightarrow \mathbb{C}(i)$ such that $Q_i[f] = f_1 + f_2 j$, where $j \perp i$. Moreover, the functions $f_l \in \mathfrak{B}_{p, \mathbb{C}}$; $l = 1, 2$ on \mathbb{B}_i . Furthermore, $\|f_l\|_{\mathfrak{B}_{p, \mathbb{C}}}^p \leq \|f\|_{\mathfrak{B}_{p, i}}^p$; $l = 1, 2$ and $p > 1$. Therefore, from [58, Theorem 9], it follows that for all $q, w \in \mathbb{B}_i$ in the complex plane $\mathbb{C}(i)$, one has

$$\begin{aligned} |f(q) - f(w)|^p &\leq 2^{p-1} (|f_1(q) - f_1(w)|^p + |f_2(q) - f_2(w)|^p) \\ &\leq 2^{p-1} M_p \left(\|f_1\|_{\mathfrak{B}_{p, \mathbb{C}}}^p \beta(q, w)^{\frac{p}{t}} + \|f_2\|_{\mathfrak{B}_{p, \mathbb{C}}}^p \beta(q, w)^{\frac{p}{t}} \right) \\ &\leq 2^{p-1} 2M_p \|f\|_{\mathfrak{B}_{p, i}}^p \beta_i(q, w)^{\frac{p}{t}} \\ &\leq 2^p M_p \|f\|_{\mathfrak{B}_p}^p \beta_i(q, w)^{\frac{p}{t}}. \end{aligned}$$

■

The following proposition on Besov spaces over the unit disk was proved in [58, Theorem 8] and for its proof on Bloch spaces of slice holomorphic functions one can refer to [47, Theorem 2.20].

Proposition 2.6. *Let $f \in \mathfrak{B}_p$, $p > 1$ and $\{a_n\}_{n \in \mathbb{N}} \subset \mathbb{H}$ be a sequence of quaternions such that*

$$f(q) = \sum_{n=0}^{\infty} q^n a_n \text{ for } q \in \mathbb{B}.$$

Then there exists a constant $K_p > 0$ such that

$$|a_n|^p \leq 2^p \frac{K_p}{n} \|f\|_{\mathfrak{B}_p}^p \text{ for } n \in \mathbb{N} \cup \{0\}.$$

Proof. Let $i, j \in \mathbb{S}$ be such that $j \perp i$. On applying Splitting Lemma 1.1, we can restrict f on \mathbb{B}_i such that $Q_i[f] = f_1 + f_2 j$, for some holomorphic functions $f_1, f_2 : \mathbb{B}_i \rightarrow \mathbb{C}(i)$ in the complex Besov space $\mathfrak{B}_{p, \mathbb{C}}$ on \mathbb{B}_i . Furthermore, for any $z \in \mathbb{B}_i$ and $p > 1$, we have

$$|f(z)|^p \leq 2^{p-1} (|f_1(z)|^p + |f_2(z)|^p).$$

Now for any $n \in \mathbb{N} \cup \{0\}$, let $a_{1,n}, a_{2,n} \in \mathbb{C}(i)$ such that $a_n = a_{1,n} + a_{2,n} j$. Thus we have

$$|f(z)|^p = \left| \sum_{n=0}^{\infty} z^n a_n \right|^p \leq 2^{p-1} \left(\left| \sum_{n=0}^{\infty} z^n a_{1,n} \right|^p + \left| \sum_{n=0}^{\infty} z^n a_{2,n} \right|^p \right) = 2^{p-1} (|f_1(z)|^p + |f_2(z)|^p).$$

Therefore, from [58, Theorem 8 (1)], it follows that for any $n \in \mathbb{N}$, we have

$$|a_{l,n}| \leq \frac{K_p}{n^{\frac{1}{p}}} \|f_l\|_{\mathfrak{B}_{p, \mathbb{C}}} \text{ ; } l = 1, 2,$$

and so $\|f_l\|_{\mathfrak{B}_{p,\mathbb{C}}} \leq \|f\|_{\mathfrak{B}_{p,i}}$; $l = 1, 2$. Then, one has

$$\begin{aligned} |a_n|^p &= 2^{p-1} (|a_{1,n}|^p + |a_{2,n}|^p) \\ &\leq 2^{p-1} \frac{K_p}{n} \left(\|f_1\|_{\mathfrak{B}_{p,\mathbb{C}}}^p + \|f_2\|_{\mathfrak{B}_{p,\mathbb{C}}}^p \right) \\ &\leq 2^{p-1} 2 \frac{K_p}{n} \|f\|_{\mathfrak{B}_{p,i}}^p \\ &\leq 2^p \frac{K_p}{n} \|f\|_{\mathfrak{B}_p}^p. \end{aligned}$$

■

Remark 2.7. Let $L^p(\mathbb{B}_i, d\lambda_i, \mathbb{H})$, $1 \leq p < \infty$ denote the space of quaternionic valued equivalence classes of measurable functions $g : \mathbb{B}_i \rightarrow \mathbb{H}$ such that

$$\int_{\mathbb{B}_i} |g(w)|^p d\lambda_i(w) < \infty.$$

Furthermore, for any $j \in \mathbb{S}$ with $j \perp i$ and $g = g_1 + g_2 j$ where g_1, g_2 are holomorphic functions in complex plane $\mathbb{C}(i)$, then, $g \in L^p(\mathbb{B}_i, d\lambda_i, \mathbb{H})$ if and only if $g_l \in L^p(\mathbb{B}_i, d\lambda_i, \mathbb{C}(i))$, $l = 1, 2$, the usual L^p -space of complex valued measurable functions on \mathbb{B}_i .

Now we define the bounded mean oscillation of the slice regular functions.

Definition 2.8. For any $z \in \mathbb{B}_i$, let $\Delta_i(z, r) = \{w \in \mathbb{B}_i : \beta_i(z, w) < r\} \subset \mathbb{B}_i$ for some $r > 0$, be the Euclidean disk. Let $f_{r,i}^*(z) = \frac{1}{2\pi} \int_{\Delta_i(z,r)} f(w) dA_i(w)$, for some arbitrary $i \in \mathbb{S}$. A slice regular function f is said to be in $BMO(\mathbb{B}_i)$ if

$$\sup_{z \in \mathbb{B}_i} \frac{1}{2\pi} \int_{\Delta_i(z,r)} |f(w) - f_{r,i}^*(z)|^p dA_i(w) < \infty,$$

with norm defined by

$$\|f\|_{BMO(\mathbb{B}_i)} = \sup_{z \in \mathbb{B}_i} \left(\frac{1}{2\pi} \int_{\Delta_i(z,r)} |f(w) - f_{r,i}^*(z)|^p dA_i(w) \right)^{\frac{1}{p}}.$$

We say function $f \in BMO(\mathbb{B})$ if

$$\|f\|_{BMO(\mathbb{B})} := \sup_{i \in \mathbb{S}} \|f\|_{BMO(\mathbb{B}_i)} := \sup_{i \in \mathbb{S}} \Lambda_{r,i}(f) < \infty,$$

where

$$\Lambda_{r,i}(f)(z) = \sup\{|f(z) - f(w)| : w \in \Delta_i(z, r)\} \text{ for some } i \in \mathbb{S}.$$

Proposition 2.9. Let $p > 1$ and $i, j \in \mathbb{S}$. Then $f \in BMO(\mathbb{B}_i)$ if and only if $f \in BMO(\mathbb{B}_j)$.

Proof. Let $f \in SR(\mathbb{B})$ and choose $w = x + yj \in \mathbb{B}_j$ and $z = x + yi \in \mathbb{B}_i$. Then by Representation formula, we have

$$|f(w)| = \frac{1}{2} |(1 - ji)f(z) + (1 + ji)f(\bar{z})| \leq |f(z)| + |f(\bar{z})|.$$

Therefore

$$\begin{aligned} \frac{1}{2\pi} \int_{\Delta_j(z,r)} |f(w) - f_{r,j}^*(z)|^p dA_j(w) &\leq 2^{\max\{p-1,0\}} \frac{1}{2\pi} \int_{\Delta_i(w,r)} |f(z) - f_{r,i}^*(w)|^p dA_i(z) \\ &+ 2^{\max\{p-1,0\}} \frac{1}{2\pi} \int_{\Delta_i(w,r)} |f(\bar{z}) - f_{r,i}^*(\bar{w})|^p dA_i(\bar{z}). \end{aligned}$$

On changing $\bar{z} \rightarrow z$ and $\bar{w} \rightarrow w$, we have

$$\frac{1}{2\pi} \int_{\Delta_j(z,r)} |f(w) - f_{r,j}^*(z)|^p dA_j(w) \leq 2^{\max\{p,1\}} \frac{1}{2\pi} \int_{\Delta_i(w,r)} |f(z) - f_{r,i}^*(w)|^p dA_i(z).$$

Thus, we conclude that for any $f \in BMO(\mathbb{B}_i)$ implies $f \in BMO(\mathbb{B}_j)$. Finally, on interchanging the role of i and j , we get the remaining one. \blacksquare

Proposition 2.10. *For $p > 1$ and $\alpha > -1$, let $f \in \mathfrak{B}_p$. Then $f \in BMO(\mathbb{B})$ if and only if $f \in BMO(\mathbb{B}_i)$, for some $i \in \mathbb{S}$.*

Proof. Since the direct part is obvious, so we only remains to prove the converse part. suppose $f \in BMO(\mathbb{B}_i)$, for some arbitrary imaginary unit i in \mathbb{S} . Therefore by Representation formula, we have

$$\begin{aligned} \frac{1}{2\pi} \int_{\Delta_j(z,r)} |f(w) - f_{r,j}^*(z)|^p dA_j(w) &\leq 2^{p-1} \frac{1}{2\pi} \left(\int_{\Delta_i(w,r)} |f(z) - f_{r,i}^*(w)|^p dA_i(z) \right) \\ &+ 2^{p-1} \frac{1}{2\pi} \left(\int_{\Delta_i(w,r)} |f(\bar{z}) - f_{r,i}^*(\bar{w})|^p dA_i(\bar{z}) \right). \end{aligned}$$

On taking supremum over all $z \in \mathbb{B}_i$, we have

$$\begin{aligned} \|f\|_{BMO(\mathbb{B}_j)} &\leq \sup_{z \in \Delta_i(w,r)} 2^{p-1} \frac{1}{2\pi} \left(\int_{\Delta_i(w,r)} |f(z) - f_{r,i}^*(w)|^p dA_i(z) \right) \\ &+ \sup_{z \in \Delta_i(w,r)} 2^{p-1} \frac{1}{2\pi} \left(\int_{\Delta_i(w,r)} |f(\bar{z}) - f_{r,i}^*(\bar{w})|^p dA_i(\bar{z}) \right) \\ &\leq 2^{p-1} 2 \|f\|_{BMO(\mathbb{B}_i)} < \infty. \end{aligned}$$

Since j is arbitrary, so we get the desired result. \blacksquare

By previous proposition we conclude the following inequality

$$\|f\|_{BMO(\mathbb{B}_i)}^p \leq \|f\|_{BMO(\mathbb{B})}^p \leq 2^p \|f\|_{BMO(\mathbb{B}_i)}^p.$$

Proposition 2.11. *For $p > 1$, let f be a slice regular function. Then $f \in \mathfrak{B}_p$ if and only if $\Lambda_r(f) \in L^p(\mathbb{B}_i, d\lambda_i, \mathbb{H})$, for some $i \in \mathbb{S}$.*

Proof. Suppose $f \in \mathfrak{B}_p$ implies $f \in \mathfrak{B}_{p,i}$. Let $j \in \mathbb{S}$ be such that $j \perp i$. By Splitting Lemma (1.1), we can restrict f on \mathbb{B}_i with respect to j , as $Q_i[f](z) = f_1(z) + f_2(z)j$, for some holomorphic functions $f_1, f_2 \in \mathbb{C}(i)$. If we decompose $\Lambda_r(f)$ on \mathbb{B}_i as $\Lambda_r(f) = \Lambda_{r,1}(f_1) + \Lambda_{r,2}(f_2)j$, for some complex oscillation functions $\Lambda_{r,1}(f_1)$ and $\Lambda_{r,2}(f_2)$. Then one can see directly from the complex result (see [58, Theorem 6]) and Remark 2.7 that the

functions $\Lambda_{r,l}(f_l)$; $l = 1, 2$ lie in the usual L^p -space of complex valued measurable functions on \mathbb{B}_i if and only if $\Lambda_r(f) \in L^p(\mathbb{B}_i, d\lambda_i, \mathbb{H})$.

Conversely, assume $\Lambda_r(f) \in L^p(\mathbb{B}_i, d\lambda_i, \mathbb{H})$. So we can write

$$\begin{aligned} \Lambda_{r,1}(f_1) + \Lambda_{r,2}(f_2)j &= \Lambda_r(f) \\ &= \sup_{i \in \mathbb{S}} \sup \{|f_1(z) - f_1(w)| : w \in \Delta_i(z, r) \subset \mathbb{B}_i\} \\ &+ \sup_{i \in \mathbb{S}} \sup \{|f_2(z) - f_2(w)| : w \in \Delta_i(z, r) \subset \mathbb{B}_i\}. \end{aligned}$$

This implies

$$\Lambda_{r,l}(f_l) = \sup_{i \in \mathbb{S}} \sup \{|f_l(z) - f_l(w)| : w \in \Delta_i(z, r) \subset \mathbb{B}_i\} \in L^p(\mathbb{B}_i, d\lambda_i, \mathbb{C}(i)), \text{ for } l = 1, 2.$$

Again thanks to above classical result, we conclude that both f_1 and f_2 belong to complex Besov space $\mathfrak{B}_{p,\mathbb{C}}$ on \mathbb{B}_i which is equivalent to $f \in \mathfrak{B}_{p,i}(\mathbb{B}_i)$ and so $f \in \mathfrak{B}_p(\mathbb{B})$. \blacksquare

Proposition 2.12. *For $p > 1$, let $f \in SR(\mathbb{B})$. Then $f \in \mathfrak{B}_p$ if and only if*

$$(2) \quad BMO(f) \in L^p(\mathbb{B}_i, d\lambda_i, \mathbb{H}), \text{ for some } i \in \mathbb{S}.$$

Proof. Let $f \in \mathfrak{B}_p$. Then $f \in \mathfrak{B}_{p,i}$. Let $j \in \mathbb{S}$ with $j \perp i$. According to Lemma (1.1), any $f \in SR(\mathbb{B})$ can be restricted to \mathbb{B}_i decomposes as $Q_i[f](z) = f_1(z) + f_2(z)j$, for some $z \in \mathbb{B}_i$ and holomorphic functions $f_1, f_2 \in \mathbb{B}_i$. Thus, the condition (2) holds if and only if

$$BMO(f_l) \in L^p(\mathbb{B}_i, d\lambda_i, \mathbb{C}(i)), \text{ for some } i \in \mathbb{S}, l = 1, 2.$$

Now, by [58, Theorem 7], it follows that the above condition holds if and only if f_1, f_2 lie in the complex Besov space $\mathfrak{B}_{p,\mathbb{C}}$ on \mathbb{B}_i which is same as $f \in \mathfrak{B}_{p,i}$ and so $f \in \mathfrak{B}_p$. \blacksquare

3. COMPOSITION OPERATORS ON BESOV SPACES

3.1. Boundedness and Compactness. In this section, we characterize boundedness and compactness of composition operators on Besov spaces of the slice holomorphic functions.

Definition 3.1. *Let $0 < p < \infty$ and let $\Phi : \mathbb{B} \rightarrow \mathbb{B}$ be a slice hyperholomorphic map such that $\Phi(\mathbb{B}_i) \subset \mathbb{B}_i$ for some $i \in \mathbb{S}$. Then the composition operator C_Φ on \mathfrak{B}_p on the unit ball \mathbb{B} induced by Φ is defined by*

$$C_\Phi f = f \circ_i \Phi, \text{ for all } f \in \mathfrak{B}_p.$$

Composition operators are extensively studied on various holomorphic function spaces of different domains in \mathbb{C} or \mathbb{C}^n . For a study of composition operators on spaces of holomorphic functions, one can refer to [27] and [52]. For composition operators on Besov spaces see, [11]. A study of composition operators on Hardy spaces of slice holomorphic functions is initiated in [49]. Recently, Carleson measures for Hardy and Bergman spaces in the quaternionic unit ball are characterized in [50]. In [13], Hankel operators are studied on Hardy spaces via Carleson measures in a quaternionic variables.

The following theorem characterize bounded composition operators on the slice regular Besov spaces \mathfrak{B}_p .

Theorem 3.2. *Let Φ be a slice holomorphic map on \mathbb{B} such that $\Phi(\mathbb{B}_i) \subset \mathbb{B}_i$ for some $i \in \mathbb{S}$. For all $q \in \mathbb{B}$ and $a \in \mathbb{B}_i$, let $\sigma_a(q) = (1 - qa)^*(a - q)$ be slice regular Möbius transformation, Then the composition operator C_Φ is bounded on Besov space \mathfrak{B}_p , $1 < p < \infty$ if and only if*

$$(3) \quad \sup \|C_\Phi \sigma_a\|_{\mathfrak{B}_p} < \infty.$$

Proof. Since the slice regular Möbius transformation on \mathbb{B}_i coincides with the usual one dimensional complex Möbius transformation, so assume $\sigma_a \in \mathfrak{B}_{p,i}$. Let $j \in \mathbb{S}$ with $j \perp i$. So we can write $\sigma_a = \sigma_{a,1} + \sigma_{a,2}j$, for each one dimensional complex Möbius transformation $\sigma_{a,l} \in \mathfrak{B}_{p,\mathbb{C}}$, $l = 1, 2$.

Therefore, from [10, Theorem 13], we have

$$(4) \quad \begin{aligned} \sup_{i \in \mathbb{S}} \int_{\mathbb{B}_i} \left| (1 - |z|^2) \frac{\partial C_\Phi \sigma_a}{\partial x_0}(z) \right|^p d\lambda_i(z) &\leq 2^{p-1} \sup_{i \in \mathbb{S}} \int_{\mathbb{B}_i} \left| (1 - |z|^2) \frac{\partial C_\Phi \sigma_{a,1}}{\partial x_0}(z) \right|^p d\lambda_i(z) \\ &\quad + 2^{p-1} \sup_{i \in \mathbb{S}} \int_{\mathbb{B}_i} \left| (1 - |z|^2) \frac{\partial C_\Phi \sigma_{a,2}}{\partial x_0}(z) \right|^p d\lambda_i(z) \\ &= 2^{p-1} (\|C_\Phi \sigma_{a,1}\|_{\mathfrak{B}_{p,\mathbb{C}}}^p + \|C_\Phi \sigma_{a,2}\|_{\mathfrak{B}_{p,\mathbb{C}}}^p) \\ &\leq 2^p \|C_\Phi \sigma_a\|_{\mathfrak{B}_{p,i}}^p. \end{aligned}$$

Now, let $q = x_0 + Iy \in \mathbb{B}$ for some $I \in \mathbb{S}$. Then by Theorem 1.3, it follows that

$$\left| \frac{\partial C_\Phi \sigma_a}{\partial x_0}(q) \right| = \left| \frac{1}{2}(1 - Ii) \frac{\partial C_\Phi \sigma_a}{\partial x_0}(z) + \frac{1}{2}(1 + Ii) \frac{\partial C_\Phi \sigma_a}{\partial x_0}(\bar{z}) \right|.$$

Since, as $|q| = |z| = |\bar{z}|$, on applying triangle inequality, we have

$$\left| (1 - |q|^2) \frac{\partial C_\Phi \sigma_a}{\partial x_0}(q) \right| \leq \left| (1 - |z|^2) \frac{\partial C_\Phi \sigma_a}{\partial x_0}(z) \right| + \left| (1 - |\bar{z}|^2) \frac{\partial C_\Phi \sigma_a}{\partial x_0}(\bar{z}) \right|.$$

On taking integral over \mathbb{B}_i on both sides of the above inequality and for $p > 1$, we see

$$(5) \quad \begin{aligned} \sup_{q \in \mathbb{B}} \int_{\mathbb{B}_i} \left| (1 - |q|^2) \frac{\partial C_\Phi \sigma_a}{\partial x_0}(q) \right|^p d\lambda_i(q) &\leq \sup_{i \in \mathbb{S}} \sup_{z \in \mathbb{B}_i} \int_{\mathbb{B}_i} \left| (1 - |z|^2) \frac{\partial C_\Phi \sigma_a}{\partial x_0}(z) \right|^p d\lambda_i(z) \\ &\quad + \sup_{i \in \mathbb{S}} \sup_{\bar{z} \in \mathbb{B}_i} \int_{\mathbb{B}_i} \left| (1 - |\bar{z}|^2) \frac{\partial C_\Phi \sigma_a}{\partial x_0}(\bar{z}) \right|^p d\lambda_i(\bar{z}) \\ &\leq 2 \sup_{i \in \mathbb{S}} \sup_{z \in \mathbb{B}_i} \int_{\mathbb{B}_i} \left| (1 - |z|^2) \frac{\partial C_\Phi \sigma_a}{\partial x_0}(z) \right|^p d\lambda_i(z). \end{aligned}$$

Thus, by using (4) in (5), we have

$$\begin{aligned} \sup \|C_\Phi \sigma_a\|_{\mathfrak{B}_p}^p &= \sup_{i \in \mathbb{S}} \sup_{q \in \mathbb{B}} \int_{\mathbb{B}_i} \left| (1 - |q|^2) \frac{\partial C_\Phi \sigma_a}{\partial x_0}(q) \right|^p d\lambda_i(q) \\ &\leq 2 \sup_{i \in \mathbb{S}} \sup_{z \in \mathbb{B}_i} \int_{\mathbb{B}_i} \left| (1 - |z|^2) \frac{\partial C_\Phi \sigma_a}{\partial x_0}(z) \right|^p d\lambda_i(z) \\ &\leq 2^{p+1} \|C_\Phi \sigma_a\|_{\mathfrak{B}_{p,i}}^p. \end{aligned}$$

Since C_Φ is bounded operator on the complex Besov space on \mathbb{B}_i , therefore $\|C_\Phi \sigma_a\|_{\mathfrak{B}_p}^p < \infty$. Now suppose condition (3) holds. Then by [10, Theorem 13], it holds if and only if C_Φ is bounded operator on the complex Besov space which is equivalent to the boundedness of C_Φ on $\mathfrak{B}_{p,i}$ and so C_Φ is bounded on \mathfrak{B}_p .

By using Splitting Lemma, Remark 2.2 and [54, Lemma 3.6], the proof of the following lemma follows easily.

Lemma 3.1. *For $p \geq 1$, let \mathfrak{B}_p be a slice regular Besov space on the unit ball \mathbb{B} . Then the following condition holds:*

- (1) *every slice regular bounded sequence $\{f_m\}_{m \in \mathbb{N}}$ in \mathfrak{B}_p on compact sets is uniformly bounded;*
- (2) *for any slice regular sequence $\{f_m\}_{m \in \mathbb{N}}$ in \mathfrak{B}_p such that $\|f_n\|_{\mathfrak{B}_p} \rightarrow 0$, $f_n - f_n(0) \rightarrow 0$ uniformly on the compact sets.*

The next result is essential for the proof of Proposition 3.3.

Lemma 3.2. [54, Lemma 3.7] *Let X, Y be two Banach spaces of analytic functions on the unit disk \mathbb{D} . Suppose*

- (1) *the point evaluation functionals on X are continuous;*
- (2) *the closed unit ball in X is a compact subset of X in the topology of uniform convergence on compact sets;*
- (3) *$T : X \rightarrow Y$ is continuous, where X and Y are equipped with the topology of uniform convergence on compact sets. Then T is a compact operator if and only if given a bounded sequence $\{f_n\}$ in X such that $f_n \rightarrow 0$ uniformly on compact sets, then the sequence $\{Tf_n\}$ converges to zero in the norm of Y .*

The following proposition gives the characterization for compact composition operators.

Proposition 3.3. *For $p > 1$, let \mathfrak{B}_p be a slice regular Besov space on the unit ball \mathbb{B} . Let Φ be a slice holomorphic map on \mathbb{B} such that $\Phi(\mathbb{B}_i) \subset \mathbb{B}_i$ for some $i \in \mathbb{S}$. Then $C_\Phi : \mathfrak{B}_p \rightarrow \mathfrak{B}_p$ is compact if and only if for any bounded sequence $\{f_m\}_{m \in \mathbb{N}}$ in \mathfrak{B}_p with $f_m \rightarrow 0$ as $m \rightarrow \infty$ on compact sets, $\|C_\Phi f_m\|_{\mathfrak{B}_p} \rightarrow 0$.*

Proof. The proof of the theorem is established if we prove the condition of Lemma 3.2. As a consequence of Lemma 3.1, we see that conditions (1) and (3) holds. Now, it remains to prove the condition (2). For this, let $\{f_m\}$ be a slice regular bounded sequence in \mathfrak{B}_p . Then by Lemma 3.1, $\{f_m\}$ is uniformly bounded on the compact sets. Consider $\{f_{m_k}\}$ a subsequence of $\{f_m\}$ in \mathfrak{B}_p such that f_{m_k} converges uniformly to h on the compact sets, for some $h \in SR(\mathbb{B})$. Let $j \in \mathbb{S}$ with $j \perp i$. Then by Lemma 1.1, there exist holomorphic functions $f_{1,m_k}, f_{2,m_k} : \mathbb{B}_i \rightarrow \mathbb{C}(i)$ such that $Q_i[f_{m_k}](z) = f_{1,m_k}(z) + f_{2,m_k}(z)j$, for some $z \in \mathbb{B}_i$. Furthermore, $f_{1,m_k} \rightarrow h_1$ and $f_{2,m_k} \rightarrow h_2$ uniformly on the compact sets, where $h_l \in \mathbb{C}(i), l = 1, 2$ with $Q_i[h] = h_1 + h_2j$. From Remark 2.2, we conclude that f_{1,m_k} and f_{2,m_k} belong to the complex Besov space $\mathfrak{B}_{p,\mathbb{C}}(\mathbb{B}_i)$. Thus, from [54, Lemma 3.8] and by

applying Minkowski's inequality and Fatou's Theorem, for $p > 1$, we have

$$\begin{aligned}
\left(\int_{\mathbb{B}_i} \left| \left(\frac{\partial h}{\partial x_0}(z) \right) (1 - |z|^2) \right|^p d\lambda_i(z) \right)^{\frac{1}{p}} &\leq \left(\int_{\mathbb{B}_i} 2^{p-1} |(h'_1(z) + h'_2(z)j)(1 - |z|^2)|^p d\lambda_i(z) \right)^{\frac{1}{p}} \\
&\leq \left(2^{p-1} \int_{\mathbb{B}_i} |(h'_1(z))(1 - |z|^2)|^p d\lambda_i(z) \right)^{\frac{1}{p}} \\
&\quad + \left(2^{p-1} \int_{\mathbb{B}_i} |(h'_2(z))(1 - |z|^2)|^p d\lambda_i(z) \right)^{\frac{1}{p}} \\
&= 2^{\frac{p-1}{p}} \left(\int_{\mathbb{B}_i} \lim_{k \rightarrow \infty} |f'_{1,m_k}(z)(1 - |z|^2)|^p d\lambda_i(z) \right)^{\frac{1}{p}} \\
&\quad + 2^{\frac{p-1}{p}} \left(\int_{\mathbb{B}_i} \lim_{k \rightarrow \infty} |f'_{2,m_k}(z)(1 - |z|^2)|^p d\lambda_i(z) \right)^{\frac{1}{p}} \\
&\leq 2^{\frac{p-1}{p}} \lim_{k \rightarrow \infty} \left(\int_{\mathbb{B}_i} |f'_{1,m_k}(z)(1 - |z|^2)|^p d\lambda_i(z) \right)^{\frac{1}{p}} \\
&\quad + 2^{\frac{p-1}{p}} \lim_{k \rightarrow \infty} \left(\int_{\mathbb{B}_i} |f'_{2,m_k}(z)(1 - |z|^2)|^p d\lambda_i(z) \right)^{\frac{1}{p}} \\
&= 2^{\frac{p-1}{p}} \left(\liminf_{k \rightarrow \infty} \|f_{1,m_k}\|_{\mathfrak{B}_{p,C}} + \liminf_{k \rightarrow \infty} \|f_{2,m_k}\|_{\mathfrak{B}_{p,C}} \right) \\
&\leq 2^{\frac{2p-1}{p}} \liminf_{k \rightarrow \infty} (\|f_{m_k}\|_{\mathfrak{B}_{p,i}}) \\
&< \infty.
\end{aligned}$$

■

The next result is the immediate consequence of Proposition 3.3.

Corollary 3.4. *For $1 < p < \infty$, let Φ be a slice holomorphic map such that $\Phi(\mathbb{B}_i) \subset \mathbb{B}_i$ for some $i \in \mathbb{S}$. If $\|\Phi\|_\infty < 1$, then $C_\Phi : \mathfrak{B}_p \rightarrow \mathfrak{B}_p$ is compact.*

Proof. Let $\{f_n\}$ be a bounded sequence in \mathfrak{B}_p . Then $f_n \in \mathfrak{B}_{p,i}$ such that $f_n \rightarrow 0$ uniformly on the compact subsets of \mathbb{B}_i for some $i \in \mathbb{S}$. Let $j \in \mathbb{S}$ be such that $j \perp i$. Let $f_{1,n}, f_{2,n} : \mathbb{B}_i \rightarrow \mathbb{C}(i)$ be holomorphic functions such that $Q_i[f](z) = f_{1,n}(z) + f_{2,n}(z)j$, for some $z = x_0 + iy \in \mathbb{B}_i$. By Remark 2.2, we have $f_{1,n}, f_{2,n}$ lie in the complex Besov space $\mathfrak{B}_{p,\mathbb{C}}$ on \mathbb{B}_i , where \mathbb{B}_i is identified with $\mathbb{D} \subset \mathbb{C}(i)$. Therefore,

$$\begin{aligned}
\sup_{i \in \mathbb{S}} \int_{\mathbb{B}_i} \left| (1 - |z|^2) \frac{\partial C_\Phi f_n}{\partial x_0}(z) \right|^p d\lambda_i(z) &\leq 2^{p-1} \sup_{i \in \mathbb{S}} \int_{\mathbb{B}_i} \left| (1 - |z|^2) \frac{\partial C_\Phi f_{1,n}}{\partial x_0}(z) \right|^p d\lambda_i(z) \\
&\quad + 2^{p-1} \sup_{i \in \mathbb{S}} \int_{\mathbb{B}_i} \left| (1 - |z|^2) \frac{\partial C_\Phi f_{2,n}}{\partial x_0}(z) \right|^p d\lambda_i(z) \\
&= 2^{p-1} (\|C_\Phi f_{1,n}\|_{\mathfrak{B}_{p,\mathbb{C}}}^p + \|C_\Phi f_{2,n}\|_{\mathfrak{B}_{p,\mathbb{C}}}^p) \\
(6) \quad &\leq 2^p \|C_\Phi f_n\|_{\mathfrak{B}_{p,i}}^p.
\end{aligned}$$

Now, appealing to Theorem 1.3 and the fact that $|q| = |\bar{z}| = |z|$, equation (6) and [53, Corollary 2.12], it follows that

$$\begin{aligned}
\sup_{q \in \mathbb{B}} \int_{\mathbb{B}_i} \left| (1 - |q|^2) \frac{\partial C_\Phi f_n}{\partial x_0}(q) \right|^p d\lambda_i(q) &\leq \sup_{i \in \mathbb{S}} \sup_{z \in \mathbb{B}_i} \int_{\mathbb{B}_i} \left| (1 - |z|^2) \frac{\partial C_\Phi f_n}{\partial x_0}(z) \right|^p d\lambda_i(z) \\
&+ \sup_{i \in \mathbb{S}} \sup_{\bar{z} \in \mathbb{B}_i} \int_{\mathbb{B}_i} \left| (1 - |\bar{z}|^2) \frac{\partial C_\Phi f_n}{\partial x_0}(\bar{z}) \right|^p d\lambda_i(\bar{z}) \\
&\leq 2^p \sup_{i \in \mathbb{S}} \sup_{z \in \mathbb{B}_i} \int_{\mathbb{B}_i} \left| (1 - |z|^2) \frac{\partial C_\Phi f_n}{\partial x_0}(z) \right|^p d\lambda_i(z) \\
&\leq 2^{p+1} \sup_{i \in \mathbb{S}} \sup_{z \in \mathbb{B}_i} \int_{\mathbb{B}_i} \left| \frac{\partial f_n}{\partial x_0}(\Phi(z)) \right|^p |1 - |z|^2|^p \\
&\quad \cdot \left| \frac{\partial \Phi}{\partial x_0}(z) \right|^p d\lambda_i(z) \\
&\leq 2^{p+1} \varepsilon.
\end{aligned}$$

Therefore, $\|C_\Phi f_m\|_{\mathfrak{B}_p}^p \rightarrow 0$ as $n \rightarrow \infty$. Hence the result. \blacksquare

The following proposition gives the compactness between Besov and Bloch spaces of slice regular functions.

Proposition 3.5. *For $p > 1$, let Φ be a slice holomorphic map on \mathbb{B} such that $\Phi(\mathbb{B}_i) \subset \mathbb{B}_i$, for some $i \in \mathbb{S}$. Then $C_\Phi : \mathfrak{B}_p \rightarrow \mathcal{B}$ is compact if and only if*

$$(7) \quad \|C_\Phi \sigma_a\|_{\mathcal{B}} \rightarrow 0, \text{ as } |a| \rightarrow 1,$$

where $\sigma_a(q) = (1 - q\bar{a})^* * (a - q)$, $q \in \mathbb{B}$ and \mathcal{B} is a slice regular Bloch space on the unit ball \mathbb{B} . Here \star denotes the slice regular product.

Proof. Let $\{\sigma_a : a \in \mathbb{B}\}$ be a set in \mathfrak{B}_p such that $\sigma_a - a \rightarrow 0$ as $|a| \rightarrow 1$. Suppose C_Φ is compact operator. Then by Lemma 3.3, $\{\sigma_a\}$ is a bounded set in \mathfrak{B}_p . Therefore, $\|C_\Phi \sigma_a\|_{\mathcal{B}} = 0$. Suppose condition (7) holds. Let f_m be a bounded sequence in $\mathfrak{B}_{p,i}$ such that $f_m \rightarrow 0$ uniformly on the compact sets as $m \rightarrow \infty$. We claim $C_\Phi : \mathfrak{B}_p \rightarrow \mathcal{B}$ is compact. For this, take $j \in \mathbb{S}$ with $j \perp i$. Let $f_{1,m}, f_{2,m}$ be holomorphic functions such that $Q_i[f_m] = f_{1,m}(z) + f_{2,m}(z)j$, for some $z = x_0 + iy \in \mathbb{B}_i$. By Remark 2.2, we have $f_{1,m}, f_{2,m}$ lie in the complex Besov space $\mathfrak{B}_{p,\mathbb{C}}(\mathbb{B}_i)$. Therefore, from [54, Theorem 4.1] and as $\|f_i\|_{\mathcal{B}_{p,\mathbb{C}}} \leq \|f\|_{\mathcal{B}_{p,i}}$, we have

$$\begin{aligned}
\|C_\Phi f_m\|_{\mathcal{B}} &= \sup_{i \in \mathbb{S}} \sup_{z \in \mathbb{B}_i} \left\{ (1 - |z|^2) \left| \frac{\partial C_\Phi(f_{1,m} + f_{2,m}j)}{\partial x_0}(z) \right| \right\} \\
&= \sup_{i \in \mathbb{S}} \sup_{z \in \mathbb{B}_i} \left\{ (1 - |z|^2) \left| \frac{\partial C_\Phi f_{1,m}}{\partial x_0}(z) + \frac{\partial C_\Phi f_{2,m}}{\partial x_0}(z)j \right| \right\} \\
&\leq \sup_{i \in \mathbb{S}} \sup_{z \in \mathbb{B}_i} \left\{ (1 - |z|^2) \left| \frac{\partial C_\Phi f_{1,m}}{\partial x_0}(z) \right| \right\} + \sup_{i \in \mathbb{S}} \sup_{z \in \mathbb{B}_i} \left\{ (1 - |z|^2) \left| \frac{\partial C_\Phi f_{2,m}}{\partial x_0}(z) \right| \right\} \\
&\leq 2 \sup_{i \in \mathbb{S}} \sup_{z \in \mathbb{B}_i} \left\{ (1 - |z|^2) \left| \frac{\partial C_\Phi f_m}{\partial x_0}(z) \right| \right\} \\
&= 2 \sup_{i \in \mathbb{S}} \left\{ \frac{(1 - |z|^2)}{(1 - |\Phi(z)|^2)} \left| \frac{\partial \Phi}{\partial x_0}(z) \right| \sup_{z \in \mathbb{B}_i} (1 - |\Phi(z)|^2) \left| \frac{\partial f_m}{\partial x_0}(\Phi(z)) \right| \right\} \\
&\leq 2 \sup_{i \in \mathbb{S}} \left\{ \frac{(1 - |z|^2)}{(1 - |\Phi(z)|^2)} \left| \frac{\partial \Phi}{\partial x_0}(z) \right| \right\} \|f_m\|_{\mathcal{B}_i} \\
&\leq 2 \sup_{i \in \mathbb{S}} \left\{ \frac{(1 - |z|^2)}{(1 - |\Phi(z)|^2)} \left| \frac{\partial \Phi}{\partial x_0}(z) \right| \right\} \|f_m\|_{\mathfrak{B}_{p,i}}.
\end{aligned}$$

Since $\{f_m\}$ is bounded in $\mathfrak{B}_{p,i}$, so $\|C_\Phi f_m\|_{\mathfrak{B}_{p,i}} \rightarrow 0$ as $m \rightarrow \infty$. Thus, $\|C_\Phi f_m\|_{\mathcal{B}} \rightarrow 0$ as $m \rightarrow \infty$. Hence by Lemma 3.3, $C_\Phi : \mathfrak{B}_p \rightarrow \mathcal{B}$ is compact. \blacksquare

4. ESSENTIAL NORM

In this section, we find some estimates for the essential norm of composition operators on the slice regular Besov space. Firstly, we define Carleson measure.

Definition 4.1. For $1 < p < \infty$, let \mathfrak{B}_p be a slice regular Besov space. Let μ be a \mathbb{H} -valued positive measure on \mathbb{B}_i . Then μ is said to be \mathbb{H} -valued p -Carleson measure on \mathbb{B} if there is a constant $M > 0$ such that

$$\int_{\mathbb{B}_i} \left| \frac{\partial f}{\partial x_0}(q) \right|^p d\mu(q) \leq M \|f\|_{\mathfrak{B}_p}^p,$$

for all $f \in \mathfrak{B}_p(\mathbb{B})$.

Theorem 4.2. Let $f \in SR(\mathbb{B})$. If $\mu = \mu_1 + \mu_2 j$ for some $i \in \mathbb{S}$. Then μ is \mathbb{H} -valued p -Carleson measure on the slice regular Besov space if and only if μ_1, μ_2 are p -Carleson measure on the complex Besov space $\mathfrak{B}_{p,\mathbb{C}}$, $1 < p < \infty$ in \mathbb{B}_i .

Proof. Let $j \in \mathbb{S}$ be such that $i \perp j$. Then for any $f \in \mathfrak{B}_{p,i}$ there exist holomorphic functions $f_1, f_2 : \mathbb{B}_i \rightarrow \mathbb{C}(i)$ such that $Q_i[f] = f_1(z) + f_2(z)j$, for some $z = x_0 + iy \in \mathbb{B}_i$. Now, μ is \mathbb{H} -valued p -Carleson measure on \mathfrak{B}_p if and only if μ is \mathbb{H} -valued p -Carleson measure on $\mathfrak{B}_{p,i}$ if and only if

$$\begin{aligned} \int_{\mathbb{B}_i} \left| \frac{\partial f}{\partial x_0}(q) \right|^p d\mu(q) &\leq M \|f\|_{\mathfrak{B}_{p,i}}^p \\ &\Leftrightarrow \int_{\mathbb{B}_i} \left| \frac{\partial(f_1 + f_2 j)}{\partial x_0}(q) \right|^p d(\mu_1 + \mu_2 j)(q) \\ &\leq M \|f_1 + f_2 j\|_{\mathfrak{B}_{p,i}}^p \\ &\Leftrightarrow \int_{\mathbb{B}_i} \left| \frac{\partial f_l}{\partial x_0}(q) \right|^p d\mu_l(q) \\ &\leq 2^p M \|f_l\|_{\mathfrak{B}_{p,\mathbb{C}}}^p \end{aligned}$$

if and only if μ_l , for $l = 1, 2$ is p -Carleson measure on $\mathfrak{B}_{p,\mathbb{C}}(\mathbb{B}_i)$. \blacksquare

Definition 4.3. [26, 47] The slice regular Möbius transformation σ_a for every $a \in \mathbb{B}$ is define as

$$\sigma_a(q) = (1 - qa)^{-*} * (a - q), \text{ for } q \in \mathbb{B},$$

where $*$ is slice regular product.

The slice regular Möbius transformation σ_a satisfies the following conditions:

- (i) $\sigma_a : \mathbb{B} \rightarrow \mathbb{B}$ is a bijective mapping,
- (ii) For all $z \in \mathbb{B}_i$, $\sigma_a(z) = \frac{a - z}{1 - \bar{a}z}$,

(iii) For all $q \in \mathbb{B}$, $\sigma_a(a) = 0$, $\sigma_a(0) = a$ and $\sigma_a \circ \sigma_a(q) = q$.

Now we give the definition of essential norm.

Definition 4.4. *The essential norm of a continuous linear operator T between the normed linear spaces X and Y is its distance from the compact operator K , that is*

$$\|T\|_e^{X \rightarrow Y} = \inf \{ \|T - K\|^{X \rightarrow Y} : K \text{ is compact operator} \},$$

where $\|\cdot\|^{X \rightarrow Y}$ denotes the operator norm and $\|\cdot\|_e^{X \rightarrow Y}$ is the essential norm.

Here, we give an essential norm estimate for composition operators on the slice regular Besov space \mathfrak{B}_p .

Theorem 4.5. *For $1 < p < \infty$ and $\alpha > -1$, let Φ be a slice holomorphic map such that $\Phi(\mathbb{B}_i) \subset \mathbb{B}_i$, for some $i \in \mathbb{S}$. Suppose the composition operator $C_\Phi : \mathfrak{B}_p \rightarrow \mathfrak{B}_p$ is bounded. Then there is an absolute constant $M \geq 1$ such that*

$$\lim_{|a| \rightarrow 1} \sup_{i \in \mathbb{S}} \sup_{\mathbb{B}_i} \int \frac{(1 - |a|^2)^{2+\alpha}}{|1 - \bar{a}q|^{2(2+\alpha)}} d\mu_p(q) \leq \|C_\Phi\|_e \leq M 2^p \lim_{|a| \rightarrow 1} \sup_{i \in \mathbb{S}} \sup_{\mathbb{B}_i} \int \frac{(1 - |a|^2)^{2+\alpha}}{|1 - \bar{a}q|^{2(2+\alpha)}} d\mu_p(q).$$

Proof. Let $f = \sum_{k=0}^{\infty} q^k a_k \in \mathfrak{B}_{p,i}$, for some $i \in \mathbb{S}$. For $0 < r < 1$, denote $\mathbb{B}_r = \{|z| < r\}$

in the complex plane $\mathbb{C}(i)$. Consider an operator $R_n f(q) = \sum_{k=n+1}^{\infty} q^k a_k$, for some integer n . Suppose $j \in \mathbb{S}$ with $j \perp i$. Then there exists holomorphic functions $f_1, f_2 : \mathbb{B}_i \rightarrow \mathbb{C}(i)$ such that $Q_i[f] = f_1(z) + f_2(z)j$, for some $z = x_0 + iy \in \mathbb{B}_i$. By Remark 2.2, we have $f_l = \sum_{k=0}^{\infty} q^k a_{l,k} \in \mathfrak{B}_{p,\mathbb{C}(\mathbb{B}_i)}$, and $R_{l,n} f_l(q) = \sum_{k=n+1}^{\infty} q^k a_{l,k}$, for some integer n and $l = 1, 2$.

Therefore, we have

$$\|C_\Phi\|_e \leq \lim_{n \rightarrow \infty} \inf \|C_\Phi R_n\|_{\mathfrak{B}_p}^p \leq \lim_{n \rightarrow \infty} \inf \sup_{\|f\|_{\mathfrak{B}_p} \leq 1} \|(C_\Phi R_n)f\|_{\mathfrak{B}_p}^p.$$

Now,

$$\begin{aligned} \|(C_\Phi R_n)f\|_{\mathfrak{B}_p}^p &= \left(|R_{1,n} f_1(\Phi(0))| + \sup_{i \in \mathbb{S}} \int_{\mathbb{B}_i} \left| (1 - |z|^2) \frac{\partial(C_\Phi R_{1,n})f_1}{\partial x_0}(\Phi(z)) \right|^p d\lambda_i(z) \right) \\ &+ \left(|R_{2,n} f_2(\Phi(0))| + \sup_{i \in \mathbb{S}} \int_{\mathbb{B}_i} \left| (1 - |z|^2) \frac{\partial(C_\Phi R_{2,n})f_2}{\partial x_0}(\Phi(z)) \right|^p d\lambda_i(z) \right) j. \end{aligned}$$

Let $\mu = \mu_{1,p} + \mu_{2,p}j$, where $\mu_{1,p}$ and $\mu_{2,p}$ are two p -Carleson measure on \mathbb{B}_i with the values in $\mathbb{C}(i)$. Then again thanks to Remark 2.2 and [42, Theorem 3.4], we have

$$\sup_{i \in \mathbb{S}} \int_{\mathbb{B}_i} \left| (1 - |z|^2) \frac{\partial(C_\Phi R_n)f}{\partial x_0}(\Phi(z)) \right|^p d\lambda_i(z) \leq 2^{p-1} \sup_{i \in \mathbb{S}} \int_{\mathbb{B}_i} \left| (1 - |z|^2) \frac{\partial(C_\Phi R_{1,n})f_1}{\partial x_0}(\Phi(z)) \right|^p d\lambda_i(z)$$

$$\begin{aligned}
& + 2^{p-1} \sup_{i \in \mathbb{S}} \int_{\mathbb{B}_i} \left| (1 - |z|^2) \frac{\partial(C_\Phi R_{2,n})f_2}{\partial x_0}(\Phi(z)) \right|^p d\lambda_i(z). \\
& = 2^{p-1} \sup_{i \in \mathbb{S}} \int_{\mathbb{B}_i} \left| \frac{\partial(C_\Phi R_{1,n})f_1}{\partial x_0}(q) \right|^p d\mu_{1,p}(z) \\
& + 2^{p-1} \sup_{i \in \mathbb{S}} \int_{\mathbb{B}_i} \left| \frac{\partial(C_\Phi R_{2,n})f_2}{\partial x_0}(q) \right|^p d\mu_{2,p}(z). \\
& = 2^{p-1} \left(\|(C_\Phi R_{1,n})f_1\|_{\mathfrak{B}_{p,c}}^p + \|(C_\Phi R_{2,n})f_2\|_{\mathfrak{B}_{p,c}}^p \right) \\
& \leq 2^p \|(C_\Phi R_n)f\|_{\mathfrak{B}_{p,i}}^p \\
& = 2^p \int_{\mathbb{B}_i \setminus \mathbb{B}_r} \left| \frac{\partial(C_\Phi R_n)f}{\partial x_0}(q) \right|^p d\mu_p(z) \\
& + 2^p \int_{\mathbb{B}_r} \left| \frac{\partial(C_\Phi R_n)f}{\partial x_0}(q) \right|^p d\mu_p(z).
\end{aligned}$$

Again by [42, Theorem 3.4], for some fixed r , we have

$$2^p \sup \int_{\mathbb{B}_r} \left| \frac{\partial(C_\Phi R_n)f}{\partial x_0}(q) \right|^p d\mu_p(z) \rightarrow 0 \text{ as } n \rightarrow \infty$$

and

$$\int_{\mathbb{B}_i \setminus \mathbb{B}_r} \left| \frac{\partial(C_\Phi R_n)f}{\partial x_0}(q) \right|^p d\mu_{p,r}(z) \leq C_1 C_2 \|\mu_p\|_r^*,$$

for some absolute constants C_1, C_2 and $\|\mu_p\|_r^* = \limsup_{a \rightarrow 1} \int_{\mathbb{B}_i} \left| \frac{\partial \sigma_a(q)}{\partial x_0} \right|^p d\mu_p(q)$, for $0 < r < 1$ and $p > 1$. Let $\mu_{p,r}$ be the restriction of measure μ_p to the set $\mathbb{B}_i \setminus \mathbb{B}_r$. Thus,

$$\begin{aligned}
\|C_\Phi\|_e^p & \leq \lim_{n \rightarrow \infty} \inf \sup_{\|f\|_{\mathfrak{B}_p} \leq 1} \|(C_\Phi R_n)f\|_{\mathfrak{B}_p}^p \\
& \leq 2^p C_1 C_2 \lim_{r \rightarrow 1} \|\mu_p\|_r^* \\
& = 2^p C_1 C_2 \limsup_{a \rightarrow 1} \int_{\mathbb{B}_i} \left| \frac{\partial \sigma_a(q)}{\partial x_0} \right|^p d\mu_p(q) \\
& = 2^p \limsup_{|a| \rightarrow 1} \int_{\mathbb{B}_i} \frac{(1 - |a|^2)^{2+\alpha}}{|1 - \bar{a}q|^{2(2+\alpha)}} d\mu_p(q).
\end{aligned}$$

Thus, we have the upper bound.

Now, let $\sigma_a(z) = \frac{a-z}{1-\bar{a}z}$ be the complex Möbius transformation on \mathbb{B}_i , associated with a . Clearly σ_a is bounded in $\mathfrak{B}_{p,i}$. Also $\sigma_a - a \rightarrow 0$ as $|a| \rightarrow 1$ uniformly on the compact subsets of \mathbb{B}_i and $|\sigma_a(z) - a| = |z| \frac{1 - |a|^2}{|1 - \bar{a}z|}$. Furthermore, $\|K(\sigma_a - a)\|_{\mathfrak{B}_{p,i}} \rightarrow 0$ as $|a| \rightarrow 1$ for some compact operator K on $\mathfrak{B}_{p,i}$. Therefore,

$$\begin{aligned}
\|C_\Phi\|_e^p & \geq \|C_\Phi - K\|_{\mathfrak{B}_p}^p \geq \|C_\Phi - K\|_{\mathfrak{B}_{p,i}}^p \\
& \geq \lim_{|a| \rightarrow 1} \|(C_\Phi - K)\sigma_a\|_{\mathfrak{B}_{p,i}}^p \\
& \geq \lim_{|a| \rightarrow 1} \sup \|C_\Phi \sigma_a\|_{\mathfrak{B}_{p,i}}^p - \lim_{|a| \rightarrow 1} \sup \|K \sigma_a\|_{\mathfrak{B}_{p,i}}^p \\
& = \lim_{|a| \rightarrow 1} \sup \sup_{i \in \mathbb{S}} \int_{\mathbb{B}_i} \frac{(1 - |a|^2)^{2+\alpha}}{|1 - \bar{a}q|^{2(2+\alpha)}} d\mu_p(q).
\end{aligned}$$

Hence the desired result. ■

Remark 4.6. By using Splitting Lemma 1.1 and Representation Theorem 1.2 it can be proved that the composition operators on spaces of slice holomorphic will be bounded if and only if the corresponding composition operators are bounded on classical holomorphic function spaces.

REFERENCES

- [1] K. Abu-Ghanem, D. Alpay, F. Colombo, D. P. Kimsey, I. Sabadini, Boundary interpolation for slice hyperholomorphic Schur functions, *Integral Equations Operator Theory*, **82** (2015), 223-248.
- [2] K. Abu-Ghanem, D. Alpay, F. Colombo, I. Sabadini, Gleasons problem and Schur multipliers in the multivariable quaternionic setting, *J. Math. Anal. Appl.*, **425** (2015), 1083-1096.
- [3] D. Alpay, F. Colombo, I. Sabadini, Inner product spaces and Krein spaces in the quaternionic setting in Recent Advances in Inverse Scattering, Schur Analysis and Stochastic Processes Operator Theory: Advances and Applications, **244** (2015), 33-65.
- [4] D. Alpay, F. Colombo, I. Sabadini, Slice Hyperholomorphic Schur Analysis, Quaderni Dipartimento di Matematica del Politecnico di Milano, QDD209, 2015 (Book preprint).
- [5] D. Alpay, F. Colombo, I. Sabadini, G. Salomon, Fock spaces in the slice hyperholomorphic setting, in Hypercomplex Analysis: New Perspectives and Applications, (2014), 43-59.
- [6] D. Alpay, F. Colombo, I. Lewkowicz, I. Sabadini, Realizations of slice hyperholomorphic generalized contractive and positive functions, *Milan J. Math.*, **83** (2015), 91-144.
- [7] D. Alpay, F. Colombo, I. Sabadini, Schur functions and their realizations in the Slice hyperholomorphic setting, *Integral Equations and Operator Theory*, **72**(2) (2012), 253-289.
- [8] D. Alpay, F. Colombo, I. Sabadini, Pontryagin-de Branges-Rovnyak spaces of slice hyperholomorphic functions, *J. Anal. Math.*, **121** (2013), 87-125.
- [9] L.V. Ahlfors, Complex Analysis, An introduction to the theory of analytic functions of one complex variable, Third Edition, McGraw-Hill, Inc, New York, 1979.
- [10] J. Arazy, S. D. Fisher, J. Peetre, Möbius invariant function spaces, *J. Reine Angew. Math.* **363** (1985), 110-145.
- [11] N. Arcozzi, R. Rochberg, E. Sawyer, Carleson measures for analytic Besov spaces, *Rev. Mat. Iberoamericana* **18** (2002), 443-510.
- [12] N. Arcozzi, G. Sarfatti, Invariant metrics for the quaternionic Hardy space, *J. Geom. Anal.*, **25** (2015), 2028-2059.
- [13] N. Arcozzi, G. Sarfatti, From Hankel operators to Carleson measures in a quaternionic variable, **arXiv:1407.8479**, (2014).
- [14] N. Arcozzi, G. Sarfatti, The orthogonal projection on slice functions on the quaternionic sphere, in Proceedings 30th International Colloquium on Group Theoretical Methods (F. Brackx, H. De Schepper and J. Vander Jeugt, editors), Journal of Physics: Conference Series, Vol. 597.
- [15] G. D. Birkhoff, Démonstration d'un théorème élémentaire sur les fonctions entières, *Comptes Rendus Acad. Sci., Paris*, **189** (1929), 473-475.
- [16] G. Birkhoff, J. von Neumann, The logic of quantum mechanics, *Ann. of Math.*, **37** (1936), 823-843.
- [17] F. Brackx, Delanghe, F. Sommen, Clifford Analysis, *Pitman Res. Notes in Math.* **76**, 1982.
- [18] F. Colombo, I. Sabadini, D.C. Struppa, Entire slice regular functions, **arXiv:1512.04215**, (2015).
- [19] F. Colombo, J. O. Gonzalez-Cervantes, M.E. Luna-Elizarraras, I. Sabadini, M.V. Shapiro, On two approaches to the Bergman theory for slice regular functions, *Advances in Hypercomplex Analysis*, Springer INdAM Series 1, (2013), 39-54.
- [20] F. Colombo, J. O. Gonzalez-Cervantes, I. Sabadini, The C-property for slice regular functions and applications to the Bergman space, *Complex Variables and Elliptic Equations*, **58** (2013), 1355-1372.

- [21] F. Colombo, J. O. Gonzalez-Cervantes, I. Sabadini, On slice biregular functions and isomorphisms of Bergman spaces, *Complex Variables and Elliptic Equations*, **57** (2012), 825-839.
- [22] F. Colombo, G. Gentili, I. Sabadini, A Cauchy kernel for slice regular functions, *Ann. Global Anal. Geom.*, **37** (2010), 361-378.
- [23] F. Colombo, I. Sabadini, A structure formula for slice monogenic functions and some of its consequences, *Hypercomplex analysis Trends in Mathematics* (2009), 101-114.
- [24] F. Colombo, I. Sabadini, D. C. Struppa, Slice monogenic functions, *Israel J. Math.* **171** (2009), 385-403.
- [25] F. Colombo, I. Sabadini, D. C. Struppa, An extension theorem for slice monogenic functions and some of its consequences, *Israel J. Math.* **177** (2009), 369-389.
- [26] F. Colombo, I. Sabadini, D. C. Struppa, Noncommutative functional calculus, *Theory and Applications of Slice Regular Hyperholomorphic Functions*, Progress in Mathematics V. 289, Birkhauser Basel 2011.
- [27] C. C. Cowen, B. D. MacCluer, Composition operators on spaces of analytic functions, *CRC Press*, Boca Raton, New York, 1995.
- [28] C. Chui, M. N. Parnes, Approximation by overconvergence of a power series, *J. Math. Anal. Appl.*, **36** (1971), 693-696.
- [29] Ž. Čučković, R. Zhao, Weighted composition operators between different weighted Bergman spaces and different Hardy spaces, *Illinois J. Math.* **51** (2007), 479-498.
- [30] G. Gentili, C. Stoppa, D. C. Struppa, Regular functions of a quaternionic variable, Springer Monographs in Mathematics, Springer, Berlin-Heidelberg, 2013.
- [31] G. Gentili, C. Stoppato, Power series and analyticity over the quaternions, *Math. Ann.*, **352** (2012), 1131-1131.
- [32] G. Gentili, D.C. Struppa, Lower bounds for polynomials of a quaternionic variable, *Proc. Amer. Math. Soc.*, **140**(5)(2012), 1659-1668.
- [33] G. Gentili, I. Sabadini, M. Shapiro, F. Sommen, D. C. Struppa, Advances in Hypercomplex Analysis, Springer Science & Business Media, 2012.
- [34] G. Gentili, C. Stoppato, The zero sets of slice regular functions and the open mapping theorem, Hypercomplex analysis and applications, 95107, Trends Math., Birkhäuser, Springer Basel AG, Basel, 2011.
- [35] G. Gentili, C. Stoppato, D. C. Struppa, A Phragmen-Lindelof principle for slice regular functions, *Bull. Belg. Math. Soc. Simon Stevin*, **18** (2011), 749-759.
- [36] [113] G. Gentili, I. Vignozzi, The Weierstrass factorization theorem for slice regular functions over the quaternions, *Ann. Global Anal. Geom.*, **40**(4) (2011), 435-466.
- [37] G. Gentili, C. Stoppato, The open mapping theorem for regular quaternionic functions, *Ann. Sc. Norm. Super. Pisa Cl. Sci.*, **8** (2009), 805-815.
- [38] G. Gentili, C. Stoppato, Zeros of regular functions and polynomials of a quaternionic variable, *Michigan Math. J.*, **56** (2008), 655-667.
- [39] G. Gentili, D.C. Struppa, The multiplicity of the zeros of polynomials with quaternionic coefficients, *Milan J. Math.*, **216** (2008), 1525.
- [40] G. Gentili, D. C. Struppa, A new theory of regular functions of a quaternionic variable, *Adv. Math.*, **216** (2007), 279-301.
- [41] G. Gentili, D. C. Struppa, A new approach to Cullen-regular functions of a quaternionic variable, *C. R. Math. Acad. Sci. Paris*, **342** (2006), 741-744.
- [42] S. Kumar, S. D. Sharma, On composition operators acting between Besov spaces, *Int. Journal of Math. Analysis* **3**(3), 2009, 133-143.
- [43] S. Kumar, K. Manzoor, P. Singh, Composition operators on slice regular Bloch type spaces of hyperholomorphic functions, *preprint*.
- [44] D. Luecking, A technique for characterizing Carleson measures on Bergman spaces, *Proc. Amer. Math. Soc.* **87**(4) (1983), 656-660.

- [45] K. Madigan, A. Matheson, Compact composition operators on the Bloch space, *Trans. Amer. Math. Soc.* **347** (1995), 2679–2687.
- [46] K. M. Madigan, Composition operators on analytic Lipschitz spaces, *Proc. Amer. Soc.* **119**(2) (1993), 465–473.
- [47] C. M. P. C. Villalba, F. Colombo, J. Gantner, J. O. G. Cervantes, Bloch, Besov and Dirichlet spaces of slice hyperholomorphic functions, *Complex Analysis and Operator Theory* **9**(2) 2015, 479–517.
- [48] A. M. Rodriguez, The essential norm of a composition operator on Bloch spaces, *Pacific. J. Math.* **188**(2) (1999), 339–351.
- [49] G. Ren, X. Wang, Slice Regular composition operators, *Complex Variables and Elliptic Equations* **61**(5) (2016), 682–711.
- [50] I. Sabadini, A. Saracco, Carleson measures for Hardy and Bergman spaces in the quaternionic unit ball, **arXiv:1601.03031** (2016).
- [51] G. Sarfatti, Elements of function theory in the unit ball of quaternions, Ph. D thesis, Università di Firenze, 2013.
- [52] J. H. Shapiro, Composition operators and classical function theory, Springer-Verlag, New York, 1993.
- [53] M. Tjani, Compact composition operators on some Möbius invariant Banach spaces, Ph. D Thesis, *Michigan State University*, 1996.
- [54] M. Tjani, Compact composition operators on Besov spaces, *Trans. Amer. Math. Soc.* **355**(11) (2003), 4683–4698.
- [55] H. Wulan, D. Zheng, K. Zhu, Compact composition operators on BMOA and the Bloch space, *Proc. Amer. Math. Soc.* **137** (2009), 3861–3868.
- [56] J. Xiao, Composition operators associated with Bloch-type spaces, *Complex Variables and Elliptic Equations* **46**(2) (2001), 109–121.
- [57] R. Zhao, Essential norms of composition operators between Bloch spaces, *Proc. Amer. Math. Soc.* **138**(7), (2010), 2537–2546.
- [58] K. Zhu, Analytic Besov spaces, *J. Math. Anal. Appl.* **157** (1991), 318–336 .
- [59] K. Zhu, Operator theory in function spaces, Marcel Dekker, New York, 1990.

DEPARTMENT OF MATHEMATICS, CENTRAL UNIVERSITY OF JAMMU, JAMMU 180 011, INDIA.
E-mail address: sanjaykmath@gmail.com

DEPARTMENT OF MATHEMATICS, CENTRAL UNIVERSITY OF JAMMU, JAMMU 180 011, INDIA.
E-mail address: khalidcuj14@gmail.com